

SUMMING INCLUSION MAPS BETWEEN SYMMETRIC SEQUENCE SPACES

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ABSTRACT. In 1973/74 Bennett and (independently) Carl proved that for $1 \leq u \leq 2$ the identity map $\text{id} : \ell_u \hookrightarrow \ell_2$ is absolutely $(u, 1)$ -summing, i. e., for every unconditionally summable sequence (x_n) in ℓ_u the scalar sequence $(\|x_n\|_{\ell_2})$ is contained in ℓ_u , which improved upon well-known results of Littlewood and Orlicz. The following substantial extension is our main result: For a 2-concave symmetric Banach sequence space E the identity map $\text{id} : E \hookrightarrow \ell_2$ is absolutely $(E, 1)$ -summing, i. e., for every unconditionally summable sequence (x_n) in E the scalar sequence $(\|x_n\|_{\ell_2})$ is contained in E . Various applications are given, e. g., to the theory of eigenvalue distribution of compact operators, where we show that the sequence of eigenvalues of an operator T on ℓ_2 with values in a 2-concave symmetric Banach sequence space E is a multiplier from ℓ_2 into E . Furthermore, we prove an asymptotic formula for the k -th approximation number of the identity map $\text{id} : \ell_2^n \hookrightarrow E_n$, where E_n denotes the linear span of the first n standard unit vectors in E , and apply it to Lorentz and Orlicz sequence spaces.

1. INTRODUCTION

In 1930 Littlewood [Lit30] proved that for every bilinear and continuous operator $\varphi : c_0 \times c_0 \rightarrow \mathbb{R}$ the quantity $\sum_{k,\ell=1}^{\infty} |\varphi(e_k, e_\ell)|^{4/3}$ is finite; this is equivalent to the statement that for every unconditionally summable sequence (x_n) in ℓ_1 the scalar sequence $(\|x_n\|_{\ell_{4/3}})$ is contained in $\ell_{4/3}$. Bennett [Ben73] and (independently) Carl [Car74] extended Littlewood's result in the following way: For $1 \leq u \leq v \leq 2$ and every unconditionally summable sequence (x_n) in ℓ_u the sequence $(\|x_n\|_{\ell_v})$ is contained in ℓ_r , where $1/r = 1/u - 1/v + 1/2$. Their result has useful applications in various parts of analysis—in particular, in approximation theory, as well as for the theory of eigenvalue distribution of compact operators, e. g., for $1 \leq u < 2$ every operator on ℓ_2 with values in ℓ_u has absolutely r -summable eigenvalues, where $1/r = 1/u - 1/2$.

The case $v = 2$ in the Bennett–Carl result is crucial (for the proof as well as for applications). Motivated by applications to interpolation theory (see, e. g., [Ovc88] and [MM99]), Maligranda and the second named author in [MM00] proved that for an Orlicz function φ for which the map $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function and for every unconditionally summable sequence (x_n) in the Orlicz sequence space ℓ_φ , the sequence $(\|x_n\|_{\ell_2})$ is contained in ℓ_φ . Moreover, based

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on complex interpolation, in [Mic99] various commutative and noncommutative variants were given.

These results were the starting point for the research upon which this article is based. Developing and using complex interpolation formulas for spaces of operators related to those of Kouba [Kou91], our main result is a far-reaching extension of the above results: For a 2-concave symmetric Banach sequence space E and every unconditionally summable sequence (x_n) in E , the sequence $(\|x_n\|_{\ell_2})$ is contained in E . In the language of $(E, 1)$ -summing operators (which we will recall later on) this means that the identity map $\text{id} : E \hookrightarrow \ell_2$ is $(E, 1)$ -summing. An example shows that the 2-concavity of E is not superfluous. As in the classical case, our result has some useful applications. We show that the sequence of eigenvalues of an operator T on ℓ_2 with values in a 2-concave symmetric Banach sequence space E is a multiplier from ℓ_2 into E , a result which for $E = \ell_u$, $1 \leq u \leq 2$, is well known (note that the space of multipliers from ℓ_2 into ℓ_u coincides with ℓ_r , $1/r = 1/u - 1/2$). Furthermore, we prove, for a 2-concave symmetric Banach sequence space E and $1 \leq k \leq n$, the asymptotic formula

$$a_k(\text{id} : \ell_2^n \hookrightarrow E_n) \asymp \frac{\lambda_E(n-k+1)}{(n-k+1)^{1/2}},$$

where $a_k(T)$ denotes the k -th approximation number of an operator T , E_n stands for the linear span of the first n standard unit vectors in E and $\lambda_E : \mathbb{N} \rightarrow \mathbb{R}_+$ is the fundamental function of the sequence space E , and apply it to Lorentz and Orlicz sequence spaces.

2. PRELIMINARIES

For a positive number a we denote by $\lfloor a \rfloor$ the largest integer less than or equal to a . If (a_n) and (b_n) are scalar sequences, we write $a_n \prec b_n$ whenever there is some $c > 0$ such that $a_n \leq c \cdot b_n$ for all n , and $a_n \asymp b_n$ whenever $a_n \prec b_n$ and $b_n \prec a_n$.

We use standard notation and notions from Banach space theory, as presented, e. g., in [LT77], [LT79] and [TJ89]. If E is a Banach space, then B_E is its (closed) unit ball and E' its dual space.

Throughout the paper, by a Banach sequence space we mean a real Banach lattice E , modelled on the set of positive integers \mathbb{N} , which contains an element x with $\text{supp } x = \mathbb{N}$. A Banach sequence space E is said to be maximal (resp., minimal) if the unit ball B_E is closed in the pointwise convergence topology induced by the space ω of all real sequences (resp., E is a separable space). Note that the condition E is maximal is equivalent to $E^\times = E'$, where as usual

$$E^\times := \left\{ x = (x_n) \in \omega; \sum_{n=1}^{\infty} |x_n y_n| < \infty \text{ for all } y = (y_n) \in E \right\}$$

is the Köthe dual of E . A Banach sequence space E is said to be symmetric provided that $\|(x_n)\|_E = \|(x_n^*)\|_E$, where (x_n^*) denotes the decreasing rearrangement of the sequence (x_n) . Note that E^\times is a maximal (symmetric, provided E is) Banach sequence space under the norm

$$\|x\| := \sup \left\{ \sum_{n=1}^{\infty} |x_n y_n|; \|y\|_E \leq 1 \right\}.$$

The fundamental function of a symmetric Banach sequence space E is defined by

$$\lambda_E(n) := \|\sum_{i=1}^n e_i\|_E, \quad n \in \mathbb{N};$$

throughout the paper, (e_n) will denote the standard unit vector basis in c_0 and E_n the linear span of the first n unit vectors. It is well known that any symmetric Banach sequence space E is continuously embedded in the symmetric Marcinkiewicz sequence space m_{λ_E} of all sequences $x = (x_n)$ such that

$$\|x\|_{\lambda_E} := \sup_{n \geq 1} x_n^{**} \lambda_E(n) < \infty,$$

where $x_n^{**} := \frac{1}{n} \sum_{k=1}^n x_k^*$. For the notions of p -convexity and q -concavity ($1 \leq p, q \leq \infty$) of a Banach lattice X (the associated constants are denoted by $\mathbf{M}^{(p)}(X)$ and $\mathbf{M}^{(q)}(X)$, respectively) we refer to [LT79, 1.d.3]—but since the notion of 2-concavity is crucial for our purposes, recall that a Banach sequence space E is called 2-concave if there exists a constant $C > 0$ such that for all $x_1, \dots, x_n \in E$,

$$\left(\sum_{i=1}^n \|x_i\|_E^2 \right)^{1/2} \leq C \cdot \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_E.$$

It is well known that this is equivalent to the notion of cotype 2 (see [LT79, 1.f.16]); recall that a Banach space X has cotype q ($2 \leq q < \infty$) if there is a constant $C > 0$ such that for finitely many $x_1, \dots, x_n \in X$,

$$\left(\sum_{i=1}^n \|x_i\|_X^q \right)^{1/q} \leq C \cdot \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) \cdot x_i \right\|_X^2 dt \right)^{1/2}.$$

Note that 2-concave symmetric Banach sequence spaces are separable and maximal. An important tool for our purposes are powers of sequence spaces: Let E be a (maximal) symmetric Banach sequence space and $0 < r < \infty$ such that $\mathbf{M}^{(\max(1,r))}(E) = 1$. Then

$$E^r := \{x \in \ell_\infty; |x|^{1/r} \in E\}$$

endowed with the norm

$$\|x\|_{E^r} := \| |x|^{1/r} \|_E^r, \quad x \in E^r,$$

is again a (maximal) symmetric Banach sequence space which is $1/\min(1, r)$ -convex. For two Banach sequence spaces E and F , the space of multipliers $M(E, F)$ from E into F consists of all scalar sequences $x = (x_n)$ such that the associated multiplication operator $(y_n) \mapsto (x_n y_n)$ is defined and bounded from E into F . $M(E, F)$ is a (maximal symmetric provided that E and F are) Banach sequence space equipped with the norm

$$\|x\|_{M(E,F)} := \sup\{\|xy\|_F; y \in B_E\}.$$

Note that if E is a Banach sequence space, then $M(E, \ell_1) = E^\times$. In the case where $E = \ell_2$ and F is 2-concave with $\mathbf{M}^{(2)}(F) = 1$ it can be easily seen that

$$(2.1) \quad M(\ell_2, F) = (((F^\times)^2)^\times)^{1/2}$$

holds isometrically. We will need that for any symmetric Banach sequence space $E \hookrightarrow \ell_2$ not equivalent to ℓ_2 ,

$$(2.2) \quad M(\ell_2, E) \hookrightarrow c_0.$$

In fact, for $F := M(\ell_2, E)$, by the assumption we have

$$\lim_{n \rightarrow \infty} \lambda_F(n) = \sup_n \left\| \sum_{i=1}^n e_i \right\|_F = \sup_n \|\text{id} : \ell_2^n \hookrightarrow E_n\| = \infty.$$

Since for any $x = (x_n) \in F$ the estimate $x_n^* \cdot \lambda_F(n) \leq \|x\|_F$ holds, the claim follows.

For all information on Banach operator ideals and s -numbers, see [DJT95], [Kön86], [Pie80] and [Pie87]. As usual, $\mathcal{L}(E, F)$ denotes the Banach space of all (bounded and linear) operators from E into F endowed with the operator norm $\|\cdot\|$. For an operator $T : X \rightarrow Y$ between Banach spaces, recall the definition of the k -th approximation number

$$a_k(T) := \inf\{\|T - S\|; S \in \mathcal{L}(X, Y) \text{ with } \text{rank}(S) < k\},$$

the k -th Weyl number

$$x_k(T) := \sup\{a_k(TS); S \in \mathcal{L}(\ell_2, X) \text{ with } \|S\| \leq 1\}$$

and the k -th Gelfand number

$$c_k(T) := \inf\{\|T|_G\|; G \subset X, \text{codim } G < k\}.$$

Moreover, for an s -number function s and a maximal symmetric Banach sequence space E , we denote by \mathcal{S}_E^s the Banach operator ideal of all operators T with $(s_n(T)) \in E$, endowed with the norm $\|T\|_{\mathcal{S}_E^s} := \|(s_n(T))\|_E$; on ℓ_2 and for fixed E , all these ideals coincide (isometrically)—for simplicity we then denote this space by \mathcal{S}_E .

For basic results and notation from interpolation theory, we refer to [BK91] and [BL78]. We recall that a mapping \mathcal{F} from (a subclass \mathcal{C} of) the category of all couples of Banach spaces into the category of all Banach spaces is said to be a method of interpolation (on \mathcal{C}) if for any couple $(X_0, X_1) \in \mathcal{C}$, the Banach space $\mathcal{F}(X_0, X_1)$ is intermediate with respect to (X_0, X_1) (i. e., $X_0 \cap X_1 \hookrightarrow \mathcal{F}(X_0, X_1) \hookrightarrow X_0 + X_1$), and $T : \mathcal{F}(X_0, X_1) \rightarrow \mathcal{F}(Y_0, Y_1)$ for all Banach couples $(X_0, X_1), (Y_0, Y_1) \in \mathcal{C}$ and every $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$. Here as usual the notation $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$ means that $T : X_0 + X_1 \rightarrow Y_0 + Y_1$ is a linear operator such that for $j = 0, 1$ the restriction of T to the space X_j is a bounded operator from X_j into Y_j . If additionally

$$\|T : \mathcal{F}(X_0, X_1) \rightarrow \mathcal{F}(Y_0, Y_1)\| \leq \max\{\|T : X_0 \rightarrow Y_0\|, \|T : X_1 \rightarrow Y_1\|\},$$

then \mathcal{F} is called an exact method of interpolation (on \mathcal{C}). Concrete examples of exact interpolation methods are the real method of interpolation $(\cdot, \cdot)_{\theta, p}$, $0 < \theta < 1$, $1 \leq p \leq \infty$ (see, e. g., [BL78, Chapter 3]) defined on the class of all Banach couples, and the complex method of interpolation $[\cdot, \cdot]_\theta$, $0 < \theta < 1$ (see, e. g., [BL78, Chapter 4]) defined on the class of couples of complex Banach spaces. Both methods are of power type θ , i. e., if $\mathcal{F} = (\cdot, \cdot)_{\theta, p}$ or $\mathcal{F} = [\cdot, \cdot]_\theta$, then for all $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$ we have

$$(2.3) \quad \|T : \mathcal{F}(X_0, X_1) \rightarrow \mathcal{F}(Y_0, Y_1)\| \leq \|T : X_0 \rightarrow Y_0\|^{1-\theta} \cdot \|T : X_1 \rightarrow Y_1\|^\theta.$$

In order to avoid misunderstandings, if we interpolate between real Banach spaces using the complex method of interpolation, we mean that we use any interpolation functor which is an extension of the complex method. For such a functor we use the original notation $[\cdot, \cdot]_\theta$.

In what follows we will often use the well-known fact (see, e. g., [BL78, 2.5.1]) that for any interpolation space X with respect to (X_0, X_1) there exists an exact interpolation functor \mathcal{F} such that $\mathcal{F}(X_0, X_1) = X$ up to equivalent norms. An

important class of interpolation spaces are K -spaces. Recall that an intermediate Banach space X with respect to a couple (X_0, X_1) is called a relative K -space if, whenever $x \in X$ and $y \in X_0 + X_1$ satisfy

$$K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1) \text{ for all } t > 0,$$

then it follows that $y \in X$, where

$$K(t, x; X_0, X_1) := \inf \{ \|x_0\|_{X_0} + t\|x_1\|_{X_1}; x = x_0 + x_1 \}, \quad t > 0,$$

is the Peetre K -functional.

A Banach couple (X_0, X_1) is said to be a relative Calderón couple if all interpolation spaces with respect to (X_0, X_1) are also relative K -spaces. This is equivalent to: For each pair of elements $x \in X_0 + X_1$ and $y \in X_0 + X_1$ satisfying $K(t, y; X_0, X_1) \leq K(t, x; X_0, X_1)$ for all $t > 0$, there exists an operator $T : (X_0, X_1) \rightarrow (X_0, X_1)$ such that $Tx = y$.

3. (E, p) -SUMMING OPERATORS

The following definition is a natural extension of the notion of absolutely (r, p) -summing operators. For two Banach spaces E and F we mean by $E \hookrightarrow F$ that E is contained in F , and the natural identity map is continuous; in this case we put $c_E^F := \|\text{id} : E \hookrightarrow F\|$ and $c_p^F := c_{\ell_p}^F$ whenever $\ell_p \hookrightarrow F$. If E and F are Banach sequence spaces with $\|e_n\|_E = 1$ for all n , then obviously $\ell_1 \hookrightarrow E$ and $c_1^E = 1$, as well as, $E^\times \hookrightarrow M(E, F)$ with $c_{E^\times}^{M(E, F)} = 1$.

Definition 3.1. For $1 \leq p < \infty$, let E be a Banach sequence space such that $\ell_p \hookrightarrow E$. Then an operator $T : X \rightarrow Y$ between Banach spaces X and Y is called (E, p) -summing (and we write $T \in \Pi_{E, p}$) if there exists a constant $C > 0$ such that for all $x_1, \dots, x_n \in X$,

$$\|(\|Tx_i\|_Y)_{i=1}^n\|_E \leq C \cdot c_p^E \cdot \sup_{x' \in B_{X'}} \left(\sum_{i=1}^n |\langle x', x_i \rangle|^p \right)^{1/p},$$

where in the sequel $(\xi_i)_{i=1}^n$ denotes the sequence $\sum_{i=1}^n \xi_i \cdot e_i$. We write $\pi_{E, p}(T)$ for the smallest constant C with the above property; in this way we obtain the Banach space $\Pi_{E, p}(X, Y)$, and the Banach operator ideal $(\Pi_{E, p}, \pi_{E, p})$ provided $\|e_n\| = 1$ for all n (see also [MM99]). Note that this ideal for $E = \ell_r$, $r \geq p$, coincides with the well-known Banach operator ideal $(\Pi_{r, p}, \pi_{r, p})$ of all absolutely (r, p) -summing operators.

Let us collect some observations needed later which are all modelled along classical results on (r, p) -summing operators. We start with the following simple fact that for each maximal Banach sequence space E an operator $T : X \rightarrow Y$ is (E, p) -summing if and only if the induced linear operator

$$\widehat{T} : \ell_p^w(X) \rightarrow E(Y), \quad \widehat{T}(x_n) := (Tx_n),$$

is defined (and hence bounded). In this case, $\|\widehat{T} : \ell_p^w(X) \rightarrow E(Y)\| = \pi_{E, p}(T)$ provided that $c_p^E = 1$. Here and in what follows, for a given Banach space X , $\ell_p^w(X)$ and $E(X)$ denote the Banach space of all weakly p -summable and absolutely E -summable sequences $x = (x_n)$ in X equipped with the norms

$$\|x\|_{\ell_p^w(X)} := \sup_{x' \in B_{X'}} \left(\sum_{n=1}^{\infty} |\langle x', x_n \rangle|^p \right)^{1/p}$$

and

$$\|x\|_{E(X)} := \|(\|x_n\|_X)\|_E,$$

respectively. It is well known that the Pietsch Domination Theorem implies that any p -summing operator $T : X \rightarrow Y$, $1 \leq p < \infty$ is a Dunford-Pettis operator, i. e., T transforms weakly convergent sequences into norm convergent sequences, and thus by Rosenthal's ℓ_1 -theorem it is compact whenever X does not contain a copy of ℓ_1 . In general this is not true for (r, p) -summing operators, as has been noted by Bennett [Ben73]; namely, the inclusion map $\ell_r \hookrightarrow \ell_\infty$ is $(r, 1)$ -summing for any $1 < r < \infty$, however not compact. But even in our more general case, the situation becomes more favorable for operators acting between special Banach spaces (see also Corollary 3.7).

Lemma 3.2. *Let Y be a Banach space and E a Banach sequence space. Then the following hold true:*

- (a) *If $T \in \Pi_{E,p}(\ell_{p'}, Y)$ with $1 < p < \infty$ and $\ell_p \hookrightarrow E \hookrightarrow c_0$, then T is a compact operator.*
- (b) *If $T \in \Pi_{E,1}(c_0, Y)$ with $\ell_1 \hookrightarrow E \hookrightarrow c_0$, then T is a compact operator.*

Proof. (a) Suppose T is not compact. Then T is no Dunford-Pettis operator by the reflexivity of $\ell_{p'}$. Thus there exists a sequence (x_n) in $\ell_{p'}$ such that $x_n \rightarrow 0$ weakly and $\|Tx_n\|_Y \geq C$ for all n with some constant $C > 0$. Consequently, $Tx_n \rightarrow 0$ weakly in Y and $\|x_n\|_{\ell_{p'}} \geq C/\|T\|$. Passing to a subsequence, we may assume by the Bessaga-Pełczyński Selection Theorem that (x_n) is equivalent to a block basis of the unit vector basis in $\ell_{p'}$ and thus to the unit vector basis in $\ell_{p'}$. Then (x_n) is weakly p -summable in $\ell_{p'}$, since clearly (e_n) is. But $T : \ell_{p'} \rightarrow Y$ is (E, p) -summing; hence $(\|Tx_n\|_Y) \in E$, and in particular $(\|Tx_n\|_Y) \in c_0$, a contradiction. For (b) we similarly show that $T : c_0 \rightarrow Y$ is a Dunford-Pettis operator, and thus compact since c_0 does not contain a copy of ℓ_1 . \square

In the following three lemmas we fix $1 \leq p < \infty$, and E will always be a Banach sequence space such that $\ell_p \hookrightarrow E$.

Lemma 3.3. *For an operator $T : X \rightarrow Y$ between Banach spaces and any $C > 0$, the following are equivalent:*

- (a) *$T \in \Pi_{E,p}$, and $\pi_{E,p}(T) \leq C$.*
- (b) *For all m the map $\Phi^m(T) : \mathcal{L}(\ell_{p'}^m, X) \rightarrow E_m(Y)$, $S \mapsto (TSe_i)$, has norm $\leq C$.*
- (c) *$\pi_{E,p}(TS) \leq C$ for all m and $S \in \mathcal{L}(\ell_{p'}^m, X)$ with $\|S\| \leq 1$.*

In particular, in this case,

$$(3.1) \quad \pi_{E,p}(T) = \sup_m \|\Phi^m(T)\| = \sup_m \{\pi_{E,p}(TS); \|S : \ell_{p'}^m \rightarrow X\| \leq 1\}.$$

The proof follows immediately from the definition and the standard observation that for each $S = \sum_{j=1}^m e_j \otimes x_j \in \mathcal{L}(\ell_{p'}^m, X)$,

$$\|S\| = \sup_{x' \in B_{X'}} \left(\sum_{j=1}^m |\langle x', x_j \rangle|^p \right)^{1/p}.$$

The following is an analogue of the well-known inclusion formulas (in the classical case due to Kwapien [Kwa68] and Tomczak-Jaegermann [TJ70]).

Lemma 3.4. For $1 \leq p < q < \infty$, let $1 < r < \infty$ be such that $1/r = 1/p - 1/q$. Then

$$\Pi_{E,p} \subset \Pi_{M(\ell_r,E),q},$$

and for all $T \in \Pi_{E,p}$,

$$\pi_{M(\ell_r,E),q}(T) \leq c_p^E \cdot c_q^{M(\ell_r,E)^{-1}} \cdot \pi_{E,p}(T).$$

Moreover, if X is a cotype 2 space, then for all Banach spaces Y ,

$$(3.2) \quad \Pi_{E,1}(X, Y) = \Pi_{M(\ell_2,E),2}(X, Y).$$

Proof. The first inclusion is easy: Let $T : X \rightarrow Y$ be (E, p) -summing. Then, for $x_1, \dots, x_n \in X$, by the Hölder inequality,

$$\begin{aligned} \|(\|Tx_k\|_1^n)_1^n\|_{M(\ell_r,E)} &= \sup_{(\lambda_k)_1^n \in B_{\ell_r^n}} \|(\lambda_k \cdot \|Tx_k\|_1^n)_1^n\|_E \\ &\leq \pi_{E,p}(T) \cdot c_p^E \cdot \sup_{(\lambda_k)_1^n \in B_{\ell_r^n}} \sup_{x' \in B_{X'}} \left(\sum_1^n |\langle x', \lambda_k x_k \rangle|^p \right)^{1/p} \\ &= \pi_{E,p}(T) \cdot c_p^E \cdot \sup_{x' \in B_{X'}} \left(\sum_1^n |\langle x', x_k \rangle|^q \right)^{1/q}, \end{aligned}$$

which gives the claim. The reverse inclusion in the second part follows from the upcoming Lemma 3.5: By a well-known result of Maurey, there exists a constant $C > 0$ such that for all $S \in \mathcal{L}(\ell_\infty^n, X)$ we have $\pi_2(S) \leq C \cdot \|S\|$ (see, e.g., [DF93, 31.7]). Then for $T \in \Pi_{M(\ell_2,E),2}(X, Y)$ and $S \in \mathcal{L}(\ell_\infty^n, X)$ with $\|S\| \leq 1$ we obtain, together with Lemma 3.5,

$$\pi_{E,1}(TS) \leq \pi_{M(\ell_2,E),2}(T) \cdot \pi_2(S) \leq C \cdot \pi_{M(\ell_2,E),2}(T),$$

which by Lemma 3.3 implies $T \in \Pi_{E,1}$. \square

As announced, it remains to prove the following:

Lemma 3.5. For $1 \leq p < q < \infty$, let $1 < r < \infty$ be such that $1/r = 1/p - 1/q$. Then

$$\Pi_{M(\ell_r,E),q} \circ \Pi_r \subset \Pi_{E,p}$$

and

$$\pi_{E,p}(TS) \leq \pi_{M(\ell_r,E),q}(T) \cdot \pi_r(S)$$

for $S \in \Pi_r(X, Y)$ and $T \in \Pi_{M(\ell_r,E),q}(Y, Z)$.

Proof. Let S and T be as in the lemma. By the Pietsch Domination Theorem, there exists a regular Borel probability measure μ on $B_{X'}$ such that for all $x \in X$,

$$\|Sx\| \leq \pi_r(S) \cdot \left(\int_{B_{X'}} |\langle x', x \rangle|^r d\mu(x') \right)^{1/r}.$$

Now take $0 \neq x_1, \dots, x_n \in X$ and put for $k = 1, \dots, n$,

$$x_k^0 := \left(\int_{B_{X'}} |\langle x', x_k \rangle|^p d\mu(x') \right)^{-1/r} \cdot x_k.$$

Then by the Hölder inequality (and $c_q^{M(\ell_r, E)} \leq c_p^E$),

$$\begin{aligned} \|(\|TSx_k\|)_1^n\|_E &\leq \|(\|TSx_k^0\|)_1^n\|_{M(\ell_r, E)} \cdot \left(\sum_1^n \int_{B_{X'}} |\langle x', x_k \rangle|^p d\mu(x') \right)^{1/r} \\ &\leq \pi_{M(\ell_r, E), q}(T) \cdot c_p^E \cdot \sup_{y' \in B_{Y'}} \left(\sum_1^n |\langle y', Sx_k^0 \rangle|^q \right)^{1/q} \\ &\quad \cdot \left(\sum_1^n \int_{B_{X'}} |\langle x', x_k \rangle|^p d\mu(x') \right)^{1/r}. \end{aligned}$$

Now complete the proof exactly as in [TJ70]. \square

As in the classical case of $(r, 2)$ -summing operators, the theory of $(F, 2)$ -summing operators is deeply connected to the theory of s -numbers. In our case a crucial tool is an extension of an inequality due to König, which can be proved exactly as in [Kön86, 2.a.3] (see [DMMa]).

Proposition 3.6. *Let F be a Banach sequence space such that $\ell_2 \hookrightarrow F$. Then $\Pi_{F,2}(\ell_2, Y) \subset \mathcal{S}_F^x(\ell_2, Y)$ for every Banach space Y . In particular, for all $T \in \Pi_{F,2}$ and k ,*

$$(3.3) \quad x_k(T) \leq \lambda_F(k)^{-1} \cdot c_2^F \cdot \pi_{F,2}(T).$$

The above result allows us to give a different proof of Lemma 3.2 in the case $p = 2$.

Corollary 3.7. *For any Banach space Y , any $(F, 2)$ -summing operator $T : \ell_2 \rightarrow Y$ is compact whenever $\ell_2 \hookrightarrow F \hookrightarrow c_0$.*

Proof. By Proposition 3.6, we have $\Pi_{F,2}(\ell_2, Y) \subset \mathcal{S}_F^x(\ell_2, Y) = \mathcal{S}_F^a(\ell_2, Y)$, which clearly gives the claim. \square

See Section 6 for the fact that for 2-convex F the ideal $\Pi_{F,2}$ and the unitary ideal \mathcal{S}_F coincide on Hilbert spaces.

4. $(E, 1)$ -SUMMING IDENTITY MAPS

The well-known results of Bennett [Ben73] and Carl [Car74] (proved independently) assure that for $1 \leq u \leq 2$ the identity map $\text{id} : \ell_u \hookrightarrow \ell_2$ is absolutely $(u, 1)$ -summing. In [MM00] an extension within the setting of Orlicz sequence spaces is presented.

Using interpolation theory, we prove as our main result the following proper extension:

Theorem 4.1. *Let E be a 2-concave symmetric Banach sequence space. Then the identity map $\text{id} : E \hookrightarrow \ell_2$ is $(E, 1)$ -summing. In other words, for every unconditionally summable sequence (x_n) in E the scalar sequence $(\|x_n\|_{\ell_2})$ is contained in E .*

The following lemmas are essential:

Lemma 4.2. *Let (E_0, E_1) be a relative Calderón couple of maximal symmetric Banach sequence spaces and E an interpolation space with respect to (E_0, E_1) . Then E^p for all $0 < p < 1$ is an interpolation space with respect to (E_0^p, E_1^p) .*

Proof. It is enough to show that E^p is a relative K -space with respect to (E_0^p, E_1^p) , i.e., if whenever $x \in E^p$ and $y \in E_0^p + E_1^p$ satisfy

$$K(t, y; E_0^p, E_1^p) \leq K(t, x; E_0^p, E_1^p) \text{ for all } t > 0,$$

then it follows that $y \in E^p$.

The claim follows from the well-known and easily verified equivalence for K -functionals, namely,

$$K(t, x; E_0^p, E_1^p) \asymp K(t^{1/p}, |x|^{1/p}; E_0, E_1)^p$$

for any $x \in E_0^p + E_1^p$ and $t > 0$, and the fact that E is a relative K -space with respect to (E_0, E_1) . \square

As an immediate consequence we obtain

Lemma 4.3. *Let E be a maximal symmetric Banach sequence space.*

- (a) *If E is 2-convex, then it is an interpolation space with respect to the couple (ℓ_2, ℓ_∞) , i. e., there exists an exact interpolation functor \mathcal{F} such that $E = \mathcal{F}(\ell_2, \ell_\infty)$.*
- (b) *If E is 2-concave, then $M(\ell_2, E)$ is 2-convex. In particular, $M(\ell_2, E)$ is an interpolation space with respect to (ℓ_2, ℓ_∞) .*

Proof. (a) Without loss of generality, we may assume that $\mathbf{M}^{(2)}(E) = 1$. Then E^2 is a maximal symmetric Banach sequence space, and by Mitiagin [Mit65] (see also [Kön86, 1.b.10]) this implies that E^2 is an interpolation space with respect to (ℓ_1, ℓ_∞) . The claim now follows by the preceding lemma and the fact that (ℓ_1, ℓ_∞) is a relative Calderón couple (see, e. g., [BK91, 2.6.9]).

(b) Without loss of generality, we may assume that $\mathbf{M}_{(2)}(E) = 1$. Then E^\times is 2-convex with $\mathbf{M}^{(2)}(E) = 1$; hence $(E^\times)^2$ and therefore also $((E^\times)^2)^\times$ are normed. Consequently, $M(\ell_2, E) = (((E^\times)^2)^\times)^{1/2}$ is 2-convex. \square

For the sake of completeness, we give a proof of the following easy and well-known result:

Lemma 4.4. *Let E and F be Banach sequence spaces, X a Banach space and \mathcal{F} an exact interpolation functor. Then*

$$\|\text{id} : \mathcal{F}(E_n(X), F_n(X)) \hookrightarrow \mathcal{F}(E_n, F_n)(X)\| \leq 1.$$

Proof. For any given $x_1, \dots, x_n \in X$, let $x'_1, \dots, x'_n \in X'$ be such that $\|x'_i\| = 1$ and $\langle x'_i, x_i \rangle = \|x_i\|$. Then for $T : \mathbb{R}^n(X) \rightarrow \mathbb{R}^n$ defined by $T((y_i)_1^n) := (\langle x'_i, y_i \rangle)_1^n$ we obviously have $\|T : E_n(X) \rightarrow E_n\| \leq 1$ and $\|T : F_n(X) \rightarrow F_n\| \leq 1$; hence,

$$\|T : \mathcal{F}(E_n(X), F_n(X)) \rightarrow \mathcal{F}(E_n, F_n)\| \leq 1.$$

Thus,

$$\begin{aligned} \|(x_i)_1^n\|_{\mathcal{F}(E_n, F_n)(X)} &= \|(\|x_i\|)_1^n\|_{\mathcal{F}(E_n, F_n)} \\ &= \|(\langle x'_i, x_i \rangle)_1^n\|_{\mathcal{F}(E_n, F_n)} \\ &\leq \|(x_i)_1^n\|_{\mathcal{F}(E_n(X), F_n(X))}. \end{aligned}$$

\square

The following lemma partially extends (in the lattice case) results of Pisier and Kouba on the complex interpolation of spaces of operators (see [Kou91], [Pi90] and also [DM00]). Recall that $\ell_2 = M(\ell_2, \ell_1)$ and $\ell_\infty = M(\ell_2, \ell_2)$; then the statement below says that, under the given assumption, the interpolation property of the spaces of multipliers (diagonal operators) can be transferred into the corresponding interpolation property of the associated spaces of bounded operators (at least in the finite-dimensional case). Note that a formula for the reverse inclusion holds whenever $\mathcal{F}(\ell_1, \ell_2) \hookrightarrow E$.

Lemma 4.5. *For a 2-concave symmetric Banach sequence space E , let \mathcal{F} be an exact interpolation functor such that $M(\ell_2, E) \hookrightarrow \mathcal{F}(\ell_2, \ell_\infty)$. Then*

$$(4.1) \quad \sup_{m,n} \|\text{id} : \mathcal{L}(\ell_2^m, E_n) \hookrightarrow \mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n))\| \leq \sqrt{2} \cdot c_{M(\ell_2, E)}^{\mathcal{F}(\ell_2, \ell_\infty)} \cdot \mathbf{M}_{(2)}(E).$$

Proof. Let $T \in \mathcal{L}(\ell_2^m, E_n)$. By a variant of the Maurey–Rosenthal Factorization Theorem (see [Def01, 4.2] and also [LPP91]) there exist an operator $R \in \mathcal{L}(\ell_2^m, \ell_2^n)$ and $\lambda \in \mathbb{R}^n$ such that

$$\|R\| \cdot \|\lambda\|_{M(\ell_2^n, E_n)} \leq \sqrt{2} \cdot \mathbf{M}_{(2)}(E) \cdot \|T\|$$

and T factors as follows:

$$\begin{array}{ccc} \ell_2^m & \xrightarrow{T} & E_n \\ & \searrow R \quad \nearrow M_\lambda & \\ & \ell_2^n & \end{array}$$

Obviously the map Φ defined by

$$\Phi(\mu) := M_\mu \circ R, \quad \mu \in \mathbb{R}^n,$$

maps the couple $(\ell_2^n, \ell_\infty^n)$ into the couple $(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n))$ such that both restrictions have norm less than or equal to $\|R\|$. Hence, by the interpolation property and the assumption, the restriction map

$$\Phi : M(\ell_2^n, E_n) \rightarrow \mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n))$$

has norm $\leq c_{M(\ell_2, E)}^{\mathcal{F}(\ell_2, \ell_\infty)} \cdot \|R\|$. Thus we obtain

$$\begin{aligned} \|T\|_{\mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n))} &= \|M_\lambda \circ R\|_{\mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n))} \\ &\leq c_{M(\ell_2, E)}^{\mathcal{F}(\ell_2, \ell_\infty)} \cdot \|R\| \cdot \|\lambda\|_{M(\ell_2^n, E_n)} \\ &\leq \sqrt{2} \cdot c_{M(\ell_2, E)}^{\mathcal{F}(\ell_2, \ell_\infty)} \cdot \mathbf{M}_{(2)}(E) \cdot \|T\|_{\mathcal{L}(\ell_2^m, E_n)}. \end{aligned}$$

□

Now we are ready to give a proof of Theorem 4.1: According to Lemma 4.3, let \mathcal{F} be an interpolation functor with $M(\ell_2, E) = \mathcal{F}(\ell_2, \ell_\infty)$. We consider the mapping

$$\Phi^{m,n} : (\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n)) \rightarrow (\ell_2^m(\ell_2^n), \ell_\infty^m(\ell_2^n))$$

defined by $\Phi^{m,n}(S) := (Se_i)_1^m$. By (3.1) we have

$$\sup_m \|\Phi^{m,n} : \mathcal{L}(\ell_2^m, \ell_1^n) \rightarrow \ell_2^m(\ell_2^n)\| = \pi_2(\text{id} : \ell_1^n \hookrightarrow \ell_2^n) = 1$$

and

$$\sup_m \|\Phi^{m,n} : \mathcal{L}(\ell_2^m, \ell_2^n) \rightarrow \ell_\infty^m(\ell_2^n)\| = \|\text{id} : \ell_2^n \hookrightarrow \ell_2^n\| = 1.$$

Then by the interpolation property we obtain that

$$\Phi^{m,n} : \mathcal{F}(\mathcal{L}(\ell_2^m, \ell_1^n), \mathcal{L}(\ell_2^m, \ell_2^n)) \rightarrow \mathcal{F}(\ell_2^m(\ell_2^n), \ell_\infty^m(\ell_2^n))$$

also has norm ≤ 1 . Now by the preceding lemma,

$$\|\Phi^{m,n} : \mathcal{L}(\ell_2^m, E_n) \rightarrow M(\ell_2^m, E_m)(\ell_2^n)\| \leq \sqrt{2} \cdot c_{M(\ell_2, E)}^{\mathcal{F}(\ell_2, \ell_\infty)} \cdot \mathbf{M}_{(2)}(E).$$

Hence, since $\sup_m \|\Phi^{m,n}\| = \pi_{M(\ell_2, E), 2}(\text{id} : E_n \hookrightarrow \ell_2^n)$,

$$\pi_{M(\ell_2, E), 2}(\text{id} : E_n \hookrightarrow \ell_2^n) \leq \sqrt{2} \cdot c_{M(\ell_2, E)}^{\mathcal{F}(\ell_2, \ell_\infty)} \cdot \mathbf{M}_{(2)}(E),$$

and since $\bigcup_n E_n$ is dense in E , this implies $(\text{id} : E \hookrightarrow \ell_2) \in \Pi_{M(\ell_2, E), 2}$. The final statement then follows from (3.2). \square

Theorem 4.1 is best possible in the following sense:

Corollary 4.6. *Let E and F be 2-concave symmetric Banach sequence spaces. Then*

$$(4.2) \quad \pi_{F,1}(\text{id} : E_n \hookrightarrow \ell_2^n) \asymp \|\text{id} : E_n \hookrightarrow F_n\|.$$

In particular, $\text{id} : E \hookrightarrow \ell_2$ is $(F, 1)$ -summing if and only if $E \hookrightarrow F$.

Proof. The upper estimate follows from Theorem 4.1 by factorization; for the lower estimate we may assume without loss of generality that $\mathbf{M}_{(2)}(E) = \mathbf{M}_{(2)}(F) = 1$. Observe that for $\lambda \in \mathbb{R}^n$ one has $\|\lambda\|_{M(\ell_2^n, F_n)} \leq \pi_{M(\ell_2, F), 2}(M_\lambda : \ell_2^n \rightarrow \ell_2^n)$ (simply take $x_i = e_i$ in the definition of $(M(\ell_2, F), 2)$ -summing); hence, by Lemma 3.3 and Lemma 3.4 as well as (2.1),

$$\begin{aligned} \pi_{F,1}(\text{id} : E_n \hookrightarrow \ell_2^n) &\geq \pi_{M(\ell_2, F), 2}(\text{id} : E_n \hookrightarrow \ell_2^n) \\ &= \sup_m \sup_{\|S : \ell_2^m \rightarrow E_n\| \leq 1} \pi_{M(\ell_2, F), 2}(\ell_2^m \xrightarrow{S} E_n \xrightarrow{\text{id}} \ell_2^n) \\ &\geq \sup_{\|\lambda\|_{M(\ell_2^n, E_n)} \leq 1} \pi_{M(\ell_2, F), 2}(M_\lambda : \ell_2^n \rightarrow \ell_2^n) \\ &\geq \sup_{\|\lambda\|_{M(\ell_2^n, E_n)} \leq 1} \|\lambda\|_{M(\ell_2^n, F_n)} \\ &= \|\text{id} : M(\ell_2^n, E_n) \hookrightarrow M(\ell_2^n, F_n)\| \\ &= \|\text{id} : ((E_n^\times)^\times)^{1/2} \hookrightarrow ((F_n^\times)^\times)^{1/2}\| \\ &= \|\text{id} : E_n \hookrightarrow F_n\|. \end{aligned}$$

\square

As a counterpart to Corollary 4.6, we show that in Theorem 4.1 the Hilbert space ℓ_2 is minimal in the following sense:

Corollary 4.7. *Let E and F be maximal symmetric Banach sequence spaces, where E is 2-concave. Then*

$$(4.3) \quad \pi_{E,1}(\text{id} : E_n \hookrightarrow F_n) \asymp \|\ell_2^n \hookrightarrow F_n\|.$$

In particular, $\text{id} : E \hookrightarrow F$ is $(E, 1)$ -summing if and only if $\ell_2 \hookrightarrow F$.

Proof. Again the upper estimate obviously follows by factorization from Theorem 4.1. For the lower estimate, note that by [CD97, p. 237 (3)] (which is also valid for $\lfloor n/2 \rfloor + 1$ instead of $\lfloor n/2 \rfloor$)

$$x_{\lfloor n/2 \rfloor + 1}(\text{id} : E_n \hookrightarrow F_n) \geq \frac{1}{\sqrt{2}} \cdot \frac{\|\text{id} : \ell_2^n \hookrightarrow F_n\|}{\|\text{id} : \ell_2^n \hookrightarrow E_n\|}.$$

Hence, by Lemma 3.4, (3.3) and [CD97, p. 237 (1)],

$$\begin{aligned}
\pi_{E,1}(\text{id} : E_n \hookrightarrow F_n) &\geq \pi_{M(\ell_2, E), 2}(\text{id} : E_n \hookrightarrow F_n) \\
&\geq \frac{\|\text{id} : \ell_2^n \hookrightarrow F_n\|}{\sqrt{2}} \cdot \frac{\|\text{id} : \ell_2^{\lfloor n/2 \rfloor + 1} \hookrightarrow E_{\lfloor n/2 \rfloor + 1}\|}{\|\text{id} : \ell_2^n \hookrightarrow E_n\|} \\
&\geq \frac{\|\text{id} : \ell_2^n \hookrightarrow F_n\|}{\sqrt{2}} \cdot \frac{\|\text{id} : \ell_2^{n - \lfloor n/2 \rfloor} \hookrightarrow E_{n - \lfloor n/2 \rfloor}\|}{\|\text{id} : \ell_2^n \hookrightarrow E_n\|} \\
&\geq \frac{\|\text{id} : \ell_2^n \hookrightarrow F_n\|}{\sqrt{2}} \cdot \frac{a_{\lfloor n/2 \rfloor + 1}(\text{id} : \ell_2^n \hookrightarrow E_n)}{\|\text{id} : \ell_2^n \hookrightarrow E_n\|} \\
&\geq \frac{\|\text{id} : \ell_2^n \hookrightarrow F_n\|}{2}.
\end{aligned}$$

□

We note that in general the assumption that a symmetric sequence space E is 2-concave is essential in Theorem 4.1, even in the class of Orlicz sequence spaces. This follows from the following proposition and the fact that there is an example, constructed by Kalton [Kal77] (see also [LT77, 4.c.3]), of an Orlicz sequence space ℓ_φ such that the identity map $\text{id} : \ell_\varphi \hookrightarrow \ell_2$ is not a strictly singular operator, i.e., id is an isomorphism on some infinite-dimensional closed subspace of ℓ_φ .

Proposition 4.8. *Let $E \hookrightarrow \ell_2$ be a Banach sequence space not equivalent to ℓ_2 . Then the identity map $\text{id} : E \hookrightarrow \ell_2$ is strictly singular whenever it is $(E, 1)$ -summing.*

Proof. Suppose that $\text{id} : E \hookrightarrow \ell_2$ is not strictly singular. Thus there exists an infinite-dimensional closed subspace X of E such that the restriction of id to X is an isomorphism from X into ℓ_2 . Let $P : \ell_2 \rightarrow X$ be a continuous linear projection. By assumption $\text{id} : E \hookrightarrow \ell_2$ is $(E, 1)$ -summing, and thus by Lemma 3.4 $T = \text{id} \circ P : \ell_2 \rightarrow \ell_2$ is $(M(\ell_2, E), 2)$ -summing. Since $E \neq \ell_2$, we get $M(\ell_2, E) \hookrightarrow c_0$ (see (2.2)). An application of Lemma 3.2 yields that T is compact, which contradicts the fact that T on X is the identity. □

In view of Theorem 4.1, the following trivial consequence seems to be of independent interest.

Corollary 4.9. *If E is a symmetric Banach sequence space not equivalent to ℓ_2 and such that the inclusion map $\text{id} : E \hookrightarrow \ell_2$ is not strictly singular, then E does not have cotype 2.*

Combining Proposition 4.8 and Corollary 4.9, we see that Kalton's example ℓ_φ is not 2-concave and $\text{id} : \ell_\varphi \hookrightarrow \ell_2$ is not $(\ell_\varphi, 1)$ -summing.

5. APPLICATIONS TO APPROXIMATION NUMBERS OF IDENTITY OPERATORS

Of special interest for applications (e.g., in approximation theory) are formulas for the asymptotic behavior of approximation numbers of finite-dimensional identity operators. One of the first well-known results in this direction is due to Pietsch [Pie74]: for $1 \leq k \leq n$ and $1 \leq p < q \leq \infty$,

$$(5.1) \quad a_k(\text{id} : \ell_q^n \hookrightarrow \ell_p^n) = (n - k + 1)^{1/p - 1/q}.$$

For the special case $1 \leq p < q = 2$ let us rewrite this as follows:

$$(5.2) \quad a_k(\text{id} : \ell_2^n \hookrightarrow \ell_p^n) = \frac{\lambda_{\ell_p}(n - k + 1)}{(n - k + 1)^{1/2}}.$$

Using Theorem 4.1, we show this formula—at least asymptotically—for all 2-concave symmetric Banach sequence spaces E instead of ℓ_p :

Theorem 5.1. *Let E be a 2-concave symmetric Banach sequence space. Then for all $1 \leq k \leq n$,*

$$(5.3) \quad a_k(\text{id} : \ell_2^n \hookrightarrow E_n) \asymp \frac{\lambda_E(n - k + 1)}{(n - k + 1)^{1/2}}.$$

The proof needs the special case $k = 1$, a result due to Szarek and Tomczak-Jaegermann [STJ80, Proposition 2.2]: Under the assumption of the theorem

$$(5.4) \quad a_1(\text{id} : \ell_2^n \hookrightarrow E_n) = \|\text{id} : \ell_2^n \hookrightarrow E_n\| \asymp \frac{\lambda_E(n)}{n^{1/2}}.$$

Proof. First we claim it is enough to show that

$$(5.5) \quad a_k(\text{id} : \ell_2^n \hookrightarrow E_n) \asymp \left\| \sum_1^{n-k+1} e_i \right\|_{M(\ell_2, E)}.$$

Indeed, the right-hand side in (5.5) is obviously equal to $\|\text{id} : \ell_2^{n-k+1} \hookrightarrow E_{n-k+1}\|$, and by (5.4) this is asymptotically equivalent to the right-hand side in (5.3).

The upper estimate in (5.5) is straightforward: Put $\lambda := \sum_1^{n-k+1} e_i \in \mathbb{K}^n$ and $\mu := \sum_{n-k+2}^n e_i \in \mathbb{K}^n$. Since the diagonal operator $M_\mu : \ell_2^n \rightarrow E_n$ has rank $k - 1$, we obtain

$$a_k(\text{id} : \ell_2^n \hookrightarrow E_n) \leq \|\text{id} - M_\mu\| = \|M_\lambda\| = \left\| \sum_1^{n-k+1} e_i \right\|_{M(\ell_2, E)}.$$

On the other hand, by a result of [CD92],

$$a_k(\text{id} : \ell_2^n \hookrightarrow E_n) = x_{n-k+1}(\text{id} : E_n \hookrightarrow \ell_2^n)^{-1},$$

so that the lower estimate in (5.5) follows from

$$(5.6) \quad x_k(\text{id} : E_n \hookrightarrow \ell_2^n) \prec \left\| \sum_1^k e_i \right\|_{M(\ell_2, E)}^{-1}.$$

In order to check (5.6), note that by Theorem 4.1 the identity map $\text{id} : E \hookrightarrow \ell_2$ is $(E, 1)$ -summing. Hence by Lemma 3.4, it is also $(M(\ell_2, E), 2)$ -summing, and by the generalized König inequality (3.3) we obtain

$$\begin{aligned} x_k(\text{id} : E_n \hookrightarrow \ell_2^n) &\leq \left\| \sum_1^k e_i \right\|_{M(\ell_2, E)}^{-1} \cdot \pi_{M(\ell_2, E), 2}(\text{id} : E_n \hookrightarrow \ell_2^n) \\ &\leq \left\| \sum_1^k e_i \right\|_{M(\ell_2, E)}^{-1} \cdot \pi_{M(\ell_2, E), 2}(\text{id} : E \hookrightarrow \ell_2), \end{aligned}$$

which completes the proof. \square

To illustrate formula (5.3), we consider Lorentz and Orlicz sequence spaces.

Corollary 5.2. (a) *Let $1 < p < 2$ and $1 \leq q \leq 2$. Then, for all $1 \leq k \leq n$,*

$$(5.7) \quad a_k(\text{id} : \ell_2^n \hookrightarrow \ell_{p,q}^n) \asymp (n - k + 1)^{1/p-1/2}.$$

(b) *Let $1 < p < 2$, and let w be a Lorentz sequence such that*

$$n \cdot w_n^{2/(2-p)} \asymp \sum_1^n w_i^{2/(2-p)}.$$

Then, for all $1 \leq k \leq n$,

$$(5.8) \quad a_k(\text{id} : \ell_2^n \hookrightarrow d_n(w, p)) \asymp (n - k + 1)^{1/p-1/2} \cdot w_{n-k+1}^{1/p}.$$

(c) Let φ be an Orlicz function such that the function $t \mapsto \varphi(\sqrt{t})$ is equivalent to a concave function. Then

$$(5.9) \quad a_k(\text{id} : \ell_2^n \hookrightarrow \ell_\varphi^n) \asymp \frac{(\varphi^{-1}(1/(n - k + 1)))^{-1}}{(n - k + 1)^{1/2}}.$$

Note that (a) is asymptotically the same result as for ℓ_p (see (5.1)); although $\ell_{p,q}$ is “very close” to ℓ_p , one may have expected an additional logarithmic term.

Proof. By Theorem 5.1, it is enough to ensure that all spaces considered in the corollary are 2-concave. For the Lorentz sequence spaces $\ell_{p,q}$, this is due to Creekmore [Cre81] (see, e. g., also [Def01]), for the Lorentz sequence spaces $d(w, p)$, see Reisner [Rei81], and for Orlicz sequence spaces this is contained in [Kom79]. \square

6. APPLICATIONS TO EIGENVALUES OF COMPACT OPERATORS AND UNITARY IDEALS

By Pitt’s theorem every operator T on ℓ_2 with values in ℓ_u , $1 \leq u < 2$, is compact. The original Bennett–Carl result implies (see, e. g., [Kön86, 2.b.11]) that its sequence of singular numbers is contained in ℓ_r , $1/r = 1/u - 1/2$, and by Weyl’s inequality (see, e. g., [Kön86, 1.b.9]) even its sequence of eigenvalues is contained in ℓ_r . Here and in what follows, complex Banach spaces are considered whenever we study eigenvalue distribution problems. Weyl’s inequality also holds for arbitrary maximal symmetric Banach sequence spaces: If the sequence of singular numbers of a compact operator on a Hilbert space is contained in a certain maximal symmetric sequence space F , then the same is true for its sequence of eigenvalues (see [Kön86, 1.b.10]). Together with Theorem 4.1, this implies the following extension of the result mentioned above:

Theorem 6.1. *Let $E \hookrightarrow \ell_2$ be a 2-concave symmetric Banach sequence space not equivalent to ℓ_2 , and let $T \in \mathcal{L}(\ell_2, \ell_2)$ be an operator with values in E . Then $T \in \mathcal{S}_{M(\ell_2, E)}$. In particular, its sequence of eigenvalues $(\lambda_n(T))$ is contained in $M(\ell_2, E)$.*

Proof. The assumption $E \neq \ell_2$ together with Corollary 4.6 assures that the identity operator on ℓ_2 is not contained in $\Pi_{M(\ell_2, E), 2}$, and by a result of Calkin (see [Pie87, 2.11.11]) it follows that every operator in $\Pi_{M(\ell_2, E), 2}(\ell_2, \ell_2)$ is compact; alternatively one may directly use Lemma 3.2 together with (2.2). Now by Theorem 4.1 and the ideal property, the operator $T : \ell_2 \xrightarrow{T} E \xrightarrow{\text{id}} \ell_2$ is contained in $\Pi_{M(\ell_2, E), 2}(\ell_2, \ell_2)$ and therefore compact, and by Proposition 3.6, the sequence of Weyl numbers $(x_n(T))$ is contained in $M(\ell_2, E)$. The second claim now follows by Weyl’s inequality mentioned above. \square

Next we discuss an alternative approach to Theorem 4.1 using interpolation of unitary ideals. We first illustrate our idea by considering the original result of Bennett and Carl:

Let $1 \leq u < 2$. By Lemma 3.3 and (3.2), the identity map $I_u : \ell_u \hookrightarrow \ell_2$ is absolutely $(u, 1)$ -summing whenever the composition $I_u S$ for any operator $S : \ell_2 \rightarrow$

ℓ_u is absolutely $(r, 2)$ -summing ($1/r = 1/u - 1/2$). By the (classical) Maurey–Rosenthal Factorization Theorem, there exist an operator $R \in \mathcal{L}(\ell_2, \ell_2)$ and $\lambda \in \ell_r$ such that S factors as follows:

$$\begin{array}{ccc} \ell_2 & \xrightarrow{S} & \ell_u \\ & \searrow R \quad \nearrow M_\lambda & \\ & \ell_2 & \end{array}$$

Then obviously the operator $I_u M_\lambda : \ell_2 \rightarrow \ell_2$ is contained in the Schatten r -class \mathcal{S}_r . By a result of Mitiagin (see, e.g., [DJT95, 10.3]) $\mathcal{S}_r = \Pi_{r,2}(\ell_2, \ell_2)$; hence $I_u S = I_u M_\lambda R$ is absolutely $(r, 2)$ -summing, which gives the claim. \square

Mitiagin's result and its proof are of interpolative nature. Alternatively, the inclusion $\mathcal{S}_r \subset \Pi_{r,2}(\ell_2, \ell_2)$ can be proved by complex interpolation of the border cases $\Pi_{2,2}(\ell_2, \ell_2) = \mathcal{S}_2$ and $\Pi_{\infty,2}(\ell_2, \ell_2) = \mathcal{L}(\ell_2, \ell_2) = \mathcal{S}_\infty$: For $\theta := 2/r$,

$$\mathcal{S}_r = [\mathcal{S}_2, \mathcal{S}_\infty]_\theta = [\Pi_{2,2}(\ell_2, \ell_2), \Pi_{\infty,2}(\ell_2, \ell_2)]_\theta \subset \Pi_{r,2}(\ell_2, \ell_2).$$

The starting point for our alternative approach to Theorem 4.1 now is an extension of Mitiagin's result, for which we need the following generalization of a result due to König (cf. [Kön86, 2.c.10]).

Lemma 6.2. *Let \mathcal{F} be an interpolation functor and (E_0, E_1) a pair of Banach sequence spaces with $\ell_p \hookrightarrow E_j$, $j = 0, 1$. Then for arbitrary Banach spaces X and Y , we have*

$$\mathcal{F}(\Pi_{E_0,p}(X, Y), \Pi_{E_1,p}(X, Y)) \hookrightarrow \Pi_{\mathcal{F}(E_0, E_1), p}(X, Y).$$

Proof. For fixed vectors $x_1, \dots, x_n \in X$ with $\sup_{x' \in B_{X'}} \sum_{j=1}^n |\langle x', x_j \rangle|^p \leq 1$, we define $\Phi(T) := (Tx_j)_{j=1}^n$ for $T \in \mathcal{L}(X, Y)$. Clearly

$$\Phi : (\Pi_{E_0,p}(X, Y), \Pi_{E_1,p}(X, Y)) \rightarrow (E_{0n}(Y), E_{1n}(Y))$$

with norm ≤ 1 . By interpolation and Lemma 4.4, we obtain that

$$\Phi : \mathcal{F}(\Pi_{E_0,p}(X, Y), \Pi_{E_1,p}(X, Y)) \rightarrow \mathcal{F}(E_{0n}, E_{1n})(Y)$$

with norm ≤ 1 . This yields that

$$\pi_{\mathcal{F}(E_0, E_1), p}(T) \leq \|T\|_{\mathcal{F}(\Pi_{E_0,p}(X, Y), \Pi_{E_1,p}(X, Y))}$$

for any $T \in \mathcal{F}(\Pi_{E_0,p}(X, Y), \Pi_{E_1,p}(X, Y))$. \square

The following theorem now admits the announced alternative proof of Theorem 4.1 exactly as it was done above for the original Bennett–Carl result; however, it also seems to be of independent interest.

Theorem 6.3. *Let F be a 2-convex maximal symmetric Banach sequence space. Then $\Pi_{F,2}(\ell_2, \ell_2) = \mathcal{S}_F$.*

Proof. The inclusion $\Pi_{F,2}(\ell_2, \ell_2) \hookrightarrow \mathcal{S}_F$ is contained in Proposition 3.6. For the reverse inclusion, we note that if $E \hookrightarrow c_0$ is a maximal symmetric space, then E is an interpolation space with respect to (ℓ_1, c_0) . Assume without loss of generality that $F \neq \ell_\infty$ and $\mathbf{M}_{(2)}(F) = 1$. By the symmetry of F , it follows that $F \hookrightarrow c_0$. Consequently, F^2 is an interpolation space with respect to (ℓ_1, c_0) , and thus by Lemma 4.2, F is an interpolation space with respect to (ℓ_2, c_0) (note that (ℓ_1, c_0) is a

relative Calderón couple since (ℓ_1, ℓ_∞) is). Hence there exists an exact interpolation functor \mathcal{F} such that $F = \mathcal{F}(\ell_2, c_0)$. By applying Lemma 6.2, we obtain

$$\mathcal{F}(\Pi_{\ell_2,2}(\ell_2, \ell_2), \Pi_{c_0,2}(\ell_2, \ell_2)) \hookrightarrow \Pi_{F,2}(\ell_2, \ell_2).$$

The claim now follows by the fact that $\mathcal{K}(\ell_2) \hookrightarrow \Pi_{c_0,2}(\ell_2, \ell_2)$ ($\mathcal{K}(\ell_2)$ denotes the space of compact operators on ℓ_2) and by a result on interpolation of unitary ideals due to Arazy [Ara78]: $\mathcal{S}_F = \mathcal{S}_{\mathcal{F}(\ell_2, c_0)} = \mathcal{F}(\mathcal{S}_2, \mathcal{K}(\ell_2))$. \square

Another nice application of Theorem 6.3 is the following:

Corollary 6.4. *Let F be a maximal symmetric Banach sequence space such that $\ell_2 \hookrightarrow F$. Then, for every Banach space X with $\dim X = n$,*

$$(6.1) \quad \pi_{F,2}(\text{id}_X) \geq C^{-1} \cdot \lambda_F(n),$$

where $C > 0$ is a constant depending on F only. Moreover, if F is 2-convex, then even

$$(6.2) \quad C^{-1} \cdot \lambda_F(n) \leq \pi_{F,2}(\text{id}_X) \leq C \cdot \lambda_F(n).$$

Proof. Let $G := m_{\lambda_F}$ be the Marcinkiewicz sequence space associated to F . By Proposition 3.6 and the fact that the continuous inclusion $F \hookrightarrow G$ is of norm one, we get that

$$\|(x_k(\text{id}_X))_1^n\|_G \leq c_2^F \cdot \pi_{F,2}(\text{id}_X).$$

Since G is a maximal symmetric Banach sequence space, it follows by the generalized Weyl inequality [Kön86, 2.a.8] that

$$\begin{aligned} \|\sum_1^n e_i\|_F &= \sup_{1 \leq k \leq n} \|\sum_1^k e_i\|_F \cdot 1 = \|(\lambda_k(\text{id}_X))_1^n\|_G \\ &\leq 2\sqrt{2e} \cdot \|(x_k(\text{id}_X))_1^n\|_G \leq 2\sqrt{2e} \cdot c_2^F \cdot \pi_{F,2}(\text{id}_X), \end{aligned}$$

where $\lambda_k(\text{id}_X)$ is the k -th eigenvalue of id_X . For the reverse estimate, note that for an operator $T : Y \rightarrow Z$ of rank n one has

$$\pi_{F,2}(T) = \sup\{\pi_{F,2}(TS) \mid S \in \mathcal{L}(\ell_2^n, Y), \|S\| \leq 1\}$$

(check the proof of [TJ89, 11.3 and 9.7]). Now let $S \in \mathcal{L}(\ell_2^n, X)$. Then by Theorem 6.3,

$$\pi_{F,2}(\text{id}_X \circ S) \leq \pi_{F,2}(\text{id}_{\ell_2^n}) \cdot \|S\| \leq \tilde{C} \cdot \|\sum_1^n e_i\|_F \cdot \|S\|,$$

where $\tilde{C} > 0$ is a constant depending only on F . \square

7. COMPLEX INTERPOLATION IN THE RANGE

Based on the case $v = 2$, Bennett and Carl also proved that for $1 \leq u \leq v \leq 2$, the identity operator $\text{id} : \ell_u \hookrightarrow \ell_v$ is absolutely $(r, 2)$ -summing whenever $1/r = 1/u - 1/v$. By using Theorem 4.1 and complex interpolation in the range, we obtain the following formal extension of our main result:

Proposition 7.1. *Let E be a 2-concave symmetric Banach sequence space. Then for $0 \leq \theta < 1$ the identity operator $\text{id} : E \hookrightarrow [\ell_2, E]_\theta$ is absolutely $(M(\ell_2, E)^{1-\theta}, 2)$ -summing.*

This now enables us to give an extension of the original Bennett–Carl result within the framework of Lorentz sequence spaces:

Corollary 7.2. *Let $1 < u_1 < v_1 < 2$ and $1 \leq u_2 \leq v_2 \leq 2$ be such that either $u_2 = v_2 = 2$ or $\frac{1/v_1-1/2}{1/u_1-1/2} = \frac{1/v_2-1/2}{1/u_2-1/2}$. Then*

$$(\text{id} : \ell_{u_1, u_2} \hookrightarrow \ell_{v_1, v_2}) \in \Pi_{\ell_{r_1, r_2}, 2},$$

where $1/r_1 = 1/u_1 - 1/v_1$ and $1/r_2 = 1/u_2 - 1/v_2$.

Proof. This directly follows from the preceding proposition and the fact that for $\theta := \frac{1/v_1-1/2}{1/u_1-1/2}$ by the reiteration theorem [BL78, 4.7.2] one has $[\ell_2, \ell_{u_1, u_2}]_\theta = \ell_{v_1, v_2}$; finally,

$$M(\ell_2, \ell_{u_1, u_2})^{1-\theta} = \ell_{\tilde{u}_1, \tilde{u}_2}^{1-\theta} = \ell_{r_1, r_2},$$

where $1/\tilde{u}_1 = 1/u_1 - 1/2$ and $1/\tilde{u}_2 = 1/u_2 - 1/2$. \square

For our applications of this result we need the following two statements:

Lemma 7.3. *Let F be a maximal symmetric sequence space such that $\ell_2 \hookrightarrow F$. Then, for every invertible operator $T : X \rightarrow Y$ between two n -dimensional Banach spaces, and for all $1 \leq k \leq n$,*

$$(7.1) \quad c_k(T) \geq C^{-1} \cdot \frac{\lambda_F(n-k+1)}{\pi_{F,2}(T^{-1})},$$

where $C := 2\sqrt{2}e \cdot c_2^F$.

Proof. We follow the proof of [CD97, p. 231] for the 2-summing norm. Take a subspace $M \subset X$ with $\text{codim } M < k$. Then

$$n - k + 1 \leq \dim M.$$

Hence by (6.1),

$$\left\| \sum_1^{n-k+1} e_i \right\|_F \leq \left\| \sum_1^{\dim M} e_i \right\|_F \leq C \cdot \pi_{F,2}(\text{id}_M).$$

Clearly (by the injectivity of $\Pi_{F,2}$),

$$\pi_{F,2}(\text{id}_M) = \pi_{F,2}(\text{id} : M \hookrightarrow X);$$

therefore the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\text{id}} & X \\ & \searrow T|_M & \nearrow T^{-1} \\ & & Y \end{array}$$

gives, as desired, $\left\| \sum_1^{n-k+1} e_i \right\|_F \leq \|T|_M\| \cdot C \cdot \pi_{F,2}(T^{-1})$. \square

The following two results extend (5.3) and (5.7):

Proposition 7.4. *Let E be a 2-concave symmetric Banach sequence space. Then for $0 \leq \theta < 1$ and all $1 \leq k \leq n$,*

$$(7.2) \quad a_k(\text{id} : [\ell_2^n, E_n]_\theta \hookrightarrow E_n) \asymp c_k(\text{id} : [\ell_2^n, E_n]_\theta \hookrightarrow E_n) \asymp \left(\frac{\lambda_E(n-k+1)}{(n-k+1)^{1/2}} \right)^{1-\theta}.$$

Proof. The estimate

$$c_k(\text{id} : [\ell_2^n, E_n]_\theta \hookrightarrow E_n) \succ \left(\frac{\lambda_E(n-k+1)}{(n-k+1)^{1/2}} \right)^{1-\theta}$$

follows from Proposition 7.1 and (7.1) together with (5.4). Obviously

$$\begin{aligned} c_k(\text{id} : [\ell_2^n, E_n]_\theta \hookrightarrow E_n) &\leq a_k(\text{id} : [\ell_2^n, E_n]_\theta \hookrightarrow E_n) \\ &\leq \|\text{id} : [\ell_2^{n-k+1}, E_{n-k+1}]_\theta \hookrightarrow E_{n-k+1}\|, \end{aligned}$$

and by (2.3) together with (5.4)

$$\begin{aligned} \|\text{id} : [\ell_2^{n-k+1}, E_{n-k+1}]_\theta \hookrightarrow E_{n-k+1}\| &\leq \|\text{id} : \ell_2^{n-k+1} \hookrightarrow E_{n-k+1}\|^{1-\theta} \\ &\prec \left(\frac{\lambda_E(n-k+1)}{(n-k+1)^{1/2}} \right)^{1-\theta}, \end{aligned}$$

which gives the claim. \square

Corollary 7.5. *Let $1 < u_1 < v_1 < 2$ and $1 \leq u_2 \leq v_2 \leq 2$ be such that either $u_2 = v_2 = 2$ or $\frac{1/v_1-1/2}{1/u_1-1/2} = \frac{1/v_2-1/2}{1/u_2-1/2}$. Then for $1 \leq k \leq n$,*

$$(7.3) \quad c_k(\text{id} : \ell_{v_1, v_2}^n \hookrightarrow \ell_{u_1, u_2}^n) \asymp a_k(\text{id} : \ell_{v_1, v_2}^n \hookrightarrow \ell_{u_1, u_2}^n) \asymp (n-k+1)^{1/u_1-1/v_1}.$$

Moreover, (7.3) also holds in the case $1 < u_1 < v_1 < 2$ and $1 \leq u_2 \leq 2 \leq v_2 \leq \infty$.

Proof. The first part is clear (use the preceding proposition together with what was mentioned in the proof of Corollary 7.2). The lower estimates for the second part now follow by factorization:

$$c_k(\text{id} : \ell_{v_1, v_2}^n \hookrightarrow \ell_{u_1, u_2}^n) \geq c_k(\text{id} : \ell_{v_1, 2}^n \hookrightarrow \ell_{u_1, 2}^n).$$

The upper estimates are again straightforward by real interpolation: Choose $0 < \theta < 1$ such that $1/v_1 = (1-\theta)/2 + \theta/u_1$; then (with the help of (2.3))

$$\begin{aligned} c_k(\text{id} : \ell_{v_1, v_2}^n \hookrightarrow \ell_{u_1, u_2}^n) &\leq a_k(\text{id} : \ell_{v_1, v_2}^n \hookrightarrow \ell_{u_1, u_2}^n) \\ &\leq \|\text{id} : \ell_{v_1, v_2}^{n-k+1} \hookrightarrow \ell_{u_1, u_2}^{n-k+1}\| \\ &\asymp \|\text{id} : (\ell_2^{n-k+1}, \ell_{u_1, u_2}^{n-k+1})_{\theta, v_2} \hookrightarrow \ell_{u_1, u_2}^{n-k+1}\| \\ &\leq \|\text{id} : \ell_2^{n-k+1} \hookrightarrow \ell_{u_1, u_2}^{n-k+1}\|^{1-\theta} \\ &\prec (n-k+1)^{1/u_1-1/v_1}. \end{aligned}$$

\square

Added in proof: Hinrichs [Hin] recently supplemented and extended some of our formulas on the asymptotic behaviour of approximation and Gelfand numbers of identity operators. Further recent improvements on some of the results in this article can be found in [DMM01, DMMb, DMMc].

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